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SOLUTION OF THE FIRST AND SECOND BOUNDARY-VALUE PROBLEMS OF NONSTATIONARY HEAT CONDUCTION FOR A TRIANGULAR REGION

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Exact solutions of nonstationary problems of heat conduction are constructed in explicit form for a regular triangle of height h with Dirichlet and Neumann's boundary conditions and an arbitrary initial condition having the property of triple symmetry in the region of the triangle. These very solutions remain valid also for the region of a rectangular triangle with an acute angle π/6, when there are no heat fluxes on the hypotenuse and smaller side, whereas Dirichlet or Neumann boundary conditions are prescribed on the larger side. Here, symmetry limitations are not imposed on the initial conditions.

1. The Dirichlet Problem for the Region Ω of a Regular Triangle. In [1] many exact solutions are given for nonstationary problems of heat conduction with different boundary conditions. For the region of a regular triangle a particular exact solution was obtained in [2] for the Dirichlet problem

$$U_t = a^2 \Delta U + p(t) , \tag{1.1}$$

$$U|_{t=0} = f(x, y) , \quad U|_{\Gamma} = \mu(t) , \tag{1.2}$$

where Γ is the boundary of the region Ω.

This solution has the form

$$W_1 = \mu(t) + \sum_{n=1}^{\infty} A_n \varphi_{2n} T_{2n} + \int_0^t E_1 (t - \tau, \xi_1, \xi_2) [p(\tau) - \dot{\mu}(\tau)] d\tau ;$$

$$E_1 (t, \xi_1, \xi_2) = \sum_{n=1}^{\infty} \frac{2}{\pi n} \varphi_{2n} T_{2n} ; \quad T_j = \exp - \left(\frac{aj\pi}{h} \right)^2 t ;$$

$$\varphi_n = \sin \pi n \xi_1 + \sin \pi n \xi_2 + \sin \pi n \xi_3 ; \tag{1.3}$$

$$\xi_i = (\bar{r} - \bar{r}_i) \bar{n}_i / h , \quad i = 1, 2, 3 ,$$

where $\bar{r}$ is the radius vector of an arbitrary point with coordinates (x, y) in the region Ω, $\bar{r}_i$ is the radius vector of the apices of the triangle Ω, and $\bar{n}_i$ are the unit normals to the sides of the triangle directed into Ω and having the property

$$\bar{n}_1 \bar{n}_2 = \bar{n}_1 \bar{n}_3 = \bar{n}_2 \bar{n}_3 = -\frac{1}{2} . \tag{1.4}$$

The three dimensionless variables ξ_i are interconnected by one equality:

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$$\xi_1 + \xi_2 + \xi_3 = 1 \; ; \quad 0 \le \xi_i \le 1 \, . \tag{1.5}$$

The region Ω in the variables (ξ_1, ξ_2) passes over into the region Ω_1 of the regular triangle, the equations of the sides of which have the form

$$\xi_1 = 0 \; ; \quad \xi_2 = 0 \; ; \quad \xi_1 + \xi_2 = 1 \, .$$

The coefficients A_n are determined from the integrals

$$A_n = \frac{4}{3} \iint_{|\Omega_1|} (f(x^*, y^*) - \mu(0)) \, \varphi_{2n} dS \, , \quad \iint_{|\Omega_1|} \varphi_{2n}^2 dS = \frac{3}{4} \, . \tag{1.6}$$

The function $E_1(t, \xi_1, \xi_2)$ from (1.3) satisfies the homogeneous heat-conduction equation (1.1) at $p(t)$ = 0, Dirichlet's homogeneous boundary conditions, and the single initial condition

$$E_1 |_\Gamma = 0 \; ; \quad E_1 |_{t=0} = 1 \, . \tag{1.7}$$

In constructing the solution W_1 from (1.3), use is made of an incomplete orthogonal system of functions $\{\varphi_{2n}\}$. Therefore, the solution W_1 is valid only for particular forms of initial conditions that admit expansion in terms of the base $\{\varphi_n\}$ with the expansion coefficients A_n from (1.6). To eliminate this drawback, we will introduce yet another system of the functions $\{\psi_n\}$:

$$\psi_n = \cos \pi n \frac{\xi_1}{h} + \cos \pi n \frac{\xi_2}{h} + \cos \pi n \frac{\xi_3}{h} \, , \quad n \ge 1 \, , \quad n \in N \, . \tag{1.8}$$

With the aid of the base $\{\psi_n\}$ we will construct the auxiliary function V:

$$V = \sum_{n=1}^{\infty} B_{2n-1} \, \psi_{2n-1} T_{2n-1} \; ; \tag{1.9}$$

$$T_{2n-1} = \exp - \left[\frac{a}{h} \pi (2n-1) \right]^2 t \, , \tag{1.10}$$

where B_n are as yet arbitrary constant coefficients.

From (1.9) and (1.10) we obtain that V satisfies the homogeneous equation (1.1) at $p(t)$ = 0 and also the following initial and boundary conditions:

$$V |_{t=0} = \sum_{n=1}^{\infty} B_n \psi_{2n-1} \; ; \quad V |_\Gamma = \sum_{n=1}^{\infty} B_n T_{2n-1} \, . \tag{1.11}$$

Using (1.5), we can show that all the functions ψ_{2n-1} at the boundary Γ reduce to unity.

It should be noted that even numbers $2n$ were used in the functions φ_{2n} and odd numbers $(2n - 1)$ in ψ_{2n-1}. If in (1.3) one uses the odd numbers $(2n - 1)$ at the sines or the even numbers $2n$ at the cosines in (1.8), then in both cases we will have functions such that at the boundary Γ will depend on the coordinates of the points of the boundary Γ; however, this will not allow one to obtain useful solutions. The fact that ψ_{2n-1} from (1.8) at the boundary Γ does not vanish also creates difficulties, but they can be overcome with the aid of the already obtained particular solution E_1 from (1.3).

By analogy with the form of the function W_1 from (1.3) the solution of problem (1.1)-(1.2) will be represented in the form

$$U_1 = \mu(t) + \sum_{n=1}^{\infty} [B_n T_{2n-1}(t)(\psi_{2n-1} - 1) + A_n \varphi_{2n} T_{2n}(t)] +$$

$$+ \int_0^t E_1(t-\tau, \xi_1, \xi_2) \left[p(t) - \dot{\mu}(\tau) + \sum_{n=1}^{\infty} B_n \dot{T}_{2n-1}(\tau) \right] d\tau . \tag{1.12}$$

It can easily be seen from direct verification that this solution of (1.12) satisfies Eq. (1.1) and boundary conditions from (1.2). Fulfilling the initial condition in (1.2), we arrive at the equation

$$\sum_{n=1}^{\infty} [B_n(\psi_{2n-1} - 1) + A_n \varphi_{2n}] = f(x, y) - \mu(0) . \tag{1.13}$$

The functions φ_n and ψ_n by definition in (1.3) and (1.8) are symmetric about the variables ξ_1, ξ_2, and ξ_3, i.e., with change of the variables ξ_1, ξ_2, and ξ_3 the form of the functions φ_{2n} and ψ_{2n-1} does not change. This property of symmetry in the variable ξ_1, ξ_2, and ξ_3 will be called triple symmetry. Thus, the left-hand side of equality (1.13) possesses a triple symmetry in the variables ξ_1, ξ_2, and ξ_3; therefore, the initial condition $f(x, y)$ must also have this property. Hence it follows that the proposed solution U_1 in (1.12) is applicable for the initial conditions $f(x, y)$ with triple symmetry. Then equality (1.13) can be considered as an expansion of the function $f(x, y) - \mu(0)$ in terms of the functional base $\{\varphi_{2n}\} + \{\psi_{2n-1}\}$.

We do not investigate the problem of completeness of the functional base consisting of two parts, $\{\varphi_{2n}\}$ and $\{\psi_{2n-1}\}$. It is clear that this base considerably expands the possibilities of the method considered in comparison with the base $\{\varphi_{2n}\}$. The functions $\{\varphi_{2n}\}$ are orthogonal to each other in the region Ω_1, just as ψ_{2n-1} [3], but φ_{2n} and ψ_{2n-1} are not orthogonal to each other and, moreover,

$$\iint_{\Omega_1} \varphi_{2n}^2 dS = \iint_{\Omega_1} \psi_{2n-1}^2 dS = \frac{3}{4} ;$$

$$\iint_{\Omega_1} \varphi_{2n} dS = \frac{3}{2\pi n} ; \quad \iint_{\Omega_1} \psi_{2m-1} dS = \frac{6}{(2m-1)^2 \pi^2} ; \tag{1.14}$$

$$\iint_{\Omega_1} \varphi_{2n} \psi_{2m-1} dS = \frac{6n}{\pi(4n^2 - (2m-1)^2)} ,$$

where Ω_1 is the region of the regular triangle in coordinates (ξ_1, ξ_2) with sides $\xi_1 = 0$, $\xi_2 = 0$, and $\xi_1 + \xi_2 = h$.

To find the generalized Fourier coefficients A_n and B_n, the left- and right-hand sides of equality (1.13) must successively be multiplied by φ_{2m} and ψ_{2m-1} and then be integrated over the region Ω_1. As a result, with the aid of (1.14) we come to a linear system for A_m and B_m:

$$\frac{3}{4}\pi^2 B_m - \frac{6}{(2m-1)^2} \sum_{n=1}^{\infty} B_n + \sum_{n=1}^{\infty} \frac{6\pi n A_n}{4n^2 - (2m-1)^2} = \pi^2 S_m ;$$

$$\sum_{m=1}^{\infty} B_m \frac{3(2m-1)^2}{2n(4n^2 - (2m-1)^2)} + \frac{3\pi}{4} A_n = \pi N_n , \quad m, n \in N . \tag{1.15}$$

$$S_m = \iint_{\Omega_1} [f(x, y) - \mu(0)] \psi_{2m-1} dS ; \quad N_n = \iint_{\Omega_1} [f(x, y) - \mu(0)] \varphi_{2n} dS . \tag{1.16}$$

System (1.15) contains an infinite number of unknowns and equations, and this creates certain difficulties in its solution. Therefore, in all of the sums of solutions of (1.12) and of system (1.15) we must restrict ourselves to a certain finite number of terms N_0 which is determined from a numerical experiment from the requirement of the necessary exactness of solution (1.12). Then system (1.15) transforms to a finite system of $2N_0$ equations for $2N_0$ of the unknowns A_n and B_n, where $n = 1, 2, ..., N_0$.

To simplify the obtaining of solution from the second equation of system (1.15), we can express A_n in terms of the coefficients B_m. Substituting this expression into the first equation in (1.15), we will have a closed system for B_m whose determinant is not equal to zero and, therefore, the determinant of the entire system (1.15) is also not equal to zero. Thus, we obtain the solution of system (1.15).

As a particular verification, we consider a simple example, where $\mu(t) = 0$ and $f(x, y) = 1$. In this case, the following equalities must hold:

$$W_1 = E_1\,; \quad B_m = 0\,; \quad A_m = \frac{2}{\pi m}\,; \quad S_m = N_m = 1\,. \tag{1.17}$$

Under conditions (1.17), the second equation in (1.15) is satisfied identically, and from the first equation we have

$$\sum_{n=1}^{\infty} \frac{12}{4n^2 - (2m-1)^2} = \pi^2\,; \quad m = 1, 2, \dots\,. \tag{1.18}$$

The proof of the convergence of series (1.18) is absent in the literature; therefore, we will perform the following calculations. We will extend the function $(\frac{1}{4} - \frac{x}{2})$, where $0 < x < 1$, by the segment $-1 < x < 0$ once in an odd and then in an even manner. Next, over the segment $0 < x < 1$ we will have the following two Fourier series for the same function:

$$\frac{1}{4} - \frac{x}{2} = \sum_{n=1}^{\infty} \frac{\sin 2\pi n x}{2\pi n} = \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos (2n-1)\,\pi x}{(2n-1)^2}\,. \tag{1.19}$$

Multiplying both sums of the second equality in (1.19) by $\cos\,(2m-1)\pi x$ and integrating in the limits from 0 to 1, we obtain the proof of the validity of equality (1.18).

2. Solution of Neumann's Problem. For Eq. (1.1), the boundary conditions are written in the form

$$\left.\frac{\partial U}{\partial n}\right|_{\Gamma} = v_0(t)\,, \tag{2.1}$$

where n is the inner normal to the boundary Γ of the region Ω.

To construct the solution of problem (1.1) and (2.1), we will need the functions φ_{2n-1} and ψ_{2n} defined in (1.3) and (1.8). These functions have the following properties:

$$\left.\frac{\partial \psi_{2n}}{\partial n}\right|_{\Gamma} = 0\,, \quad \left.\frac{\partial \varphi_{2n-1}}{\partial n}\right|_{\Gamma} = \frac{\pi(2n-1)}{h}\,. \tag{2.2}$$

From the definition of the variables ξ_i in (1.3) we have $\xi_i = \bar{n}_i/h$ and therefore

$$\text{grad}\,(\xi_1^2 + \xi_2^2 + \xi_3^2) = 2\,(\xi_1 \bar{n}_1 + \xi_2 \bar{n}_2 + \xi_3 \bar{n}_3)/h\,;$$

$$\Delta(\xi_1^2+\xi_2^2+\xi_3^2)=2(\bar{n}_1^2+\bar{n}_2^2+\bar{n}_3^2)=\frac{6}{h^2}; \tag{2.3}$$

$$\frac{\partial}{\partial n}(\xi_1^2+\xi_2^2+\xi_3^2)\,|_\Gamma=\bar{n}_3\,\text{grad}\,(\xi_1^2+\xi_2^2+\xi_3^2)\,|_{\xi_3=0}=$$

$$=-\frac{1}{h}(\xi_1+\xi_2)\,|_{\xi_3=0}=-\frac{1}{h}.$$

Taking into account the properties (2.3) for the quadratic sum $(\xi_1^2+\xi_2^2+\xi_3^2)$, we introduce the auxiliary function V_0 in the following way:

$$V_0=(\xi_1^2+\xi_2^2+\xi_3^2)-\frac{1}{2}+\frac{6a^2t}{h^2}.$$

From Eq. (2.3) it follows that V_0 is the solution of the problem

$$a^2\Delta V_0=V_{0t},\quad \left.\frac{\partial V_0}{\partial n}\right|_\Gamma=-\frac{1}{h}. \tag{2.4}$$

Thus, the function V_0 satisfies the homogeneous equation (1.1) and boundary condition (2.4).

We will also need the following Fourier series:

$$\frac{1}{6}-x-x^2=\sum_{n=1}^{\infty}\frac{\cos 2\pi nx}{\pi^2n^2},$$
$$\frac{1}{4}(x-x^2)=2\sum_{n=1}^{\infty}\frac{\sin(2n-1)\pi x}{\pi^3(2n-1)^3}. \tag{2.5}$$

Replacing x by ξ_1/h, ξ_2/h, and ξ_3/h in (2.5) and taking into account (1.5), we have two auxiliary equalities in the series

$$(\xi_1^2+\xi_2^2+\xi_3^2)-\frac{1}{2}=\sum_{n=1}^{\infty}\frac{\psi_{2n}}{\pi^2n^2}=\frac{1}{2}-\sum_{n=1}^{\infty}\frac{8\varphi_{2n-1}}{\pi^3(2n-1)^3}, \tag{2.6}$$

$$E_2=\sum_{n=1}^{\infty}\frac{\psi_{2n}}{\pi^2n^2}T_{2n}-(\xi_1^2+\xi_2^2+\xi_3^2)+\frac{1}{2}-\frac{6}{h^2}a^2t. \tag{2.7}$$

Using the properties (2.2) and (2.4) of the functions ψ_{2n} and V_0 and also equality (2.6), we can show that E_2 is the solution of the following problem:

$$E_{2t}=a^2\Delta E_2;\quad E_2\,|_{t=0}=0;\quad \left.\frac{\partial E_2}{\partial n}\right|_\Gamma=\frac{1}{h}. \tag{2.8}$$

In the construction of the solution of problem (1.1) and (2.1), the function E_2 plays the same role as the function E_1 in constructing the solution of the first boundary-value problem in (1.12).

Omitting the intermediate calculations, as in (1.12), we represent the solution U_2 of the second boundary-value problem in the form

$$U_2 = \int_0^t E_2 (t - \tau, \xi_1, \xi_2)\, \nu_{1t} (\tau)\, d\tau + \int_0^t p (\tau)\, d\tau +$$

$$+ \sum_{n=1}^{\infty} B_n \psi_{2n} T_{2n} + \sum_{n=1}^{\infty} A_n \varphi_{2n-1} T_{2n-1} - \tag{2.9}$$

$$- \nu_1 (0) \left[\frac{1}{h^2} (\xi_1^2 + \xi_2^2 + \xi_3^2) - \frac{1}{2} + \frac{6}{h^2} a^2 t \right],$$

$$\nu_1 (t) = h\nu_0 (t) - \pi \sum_{n=1}^{\infty} (2n - 1) A_n T_{2n-1} ; \quad \nu_1 (0) = h\nu_0 (0) - \pi \sum_{n=1}^{\infty} (2n - 1) A_n .$$

By its structure U_2 from (2.9) satisfies the inhomogeneous equation (1.1) and boundary conditions (2.1). Satisfying the initial condition from (1.2), we obtain the equation

$$\sum_{n=1}^{\infty} B_n \psi_{2n} + \sum_{n=1}^{\infty} A_n \varphi_{2n-1} + \left[\pi \sum_{n=1}^{\infty} (2n - 1) A_n - h\nu_0 (0) \right] \times$$

$$\times \left[\xi_1^2 + \xi_2^2 + \xi_3^2 - \frac{1}{2} \right] = f (x, y) . \tag{2.10}$$

The functional base $\{\varphi_{2n-1}\}$ is orthogonal in the region Ω_1, just as the base $\{\psi_{2n}\}$, but φ_{2n-1} and ψ_{2n} are not orthogonal to each other and, moreover,

$$\iint_{\Omega_1} \psi_{2n}^2 dS = \frac{3}{4} ; \quad \iint_{\Omega_1} \psi_{2n} dS = 0 ;$$

$$\iint_{\Omega_1} \varphi_{2n-1}^2 dS = \frac{3}{4} + \frac{12}{\pi^2 (2n - 1)^2} ; \quad \iint_{\Omega_1} \varphi_{2n-1} dS = \frac{3}{\pi (2n - 1)} ;$$

$$\iint_{\Omega_1} \varphi_{2n-1} \psi_{2m} dS = \frac{3 (2n - 1)}{\pi [(2n - 1)^2 - 4m^2]} ; \tag{2.11}$$

$$\iint_{\Omega_1} \varphi_{2n-1} \varphi_{2m-1} dS = 0 ; \quad \iint_{\Omega_1} \psi_{2n} \psi_{2m} dS = 0 , \quad m \neq n .$$

Just as with the solution of the first boundary-value problem, to obtain a linear system for A_n and B_n, we multiply the left- and right-hand sides of (2.10) in succession by φ_{2m-1} and ψ_{2m}.

It should be noted that in multiplying (2.10) by ψ_{2m} for greater convenience it is necessary to use expression (2.6) in terms of the functions ψ_{2n}, while in multiplying (2.10) by φ_{2m-1} this very expression must be used in terms of the functions φ_{2n-1}. As a result, using (2.11), we have the following linear system for A_n and B_n:

$$B_m + \sum_{n=1}^{\infty} \frac{4(2n-1)A_n}{\pi[(2n-1)^2 - 4m^2]} +$$

$$+ \frac{1}{\pi^2 m^2}\left[\sum_{n=1}^{\infty} \pi(2n-1)A_n - h\nu_0(0)\right] = \frac{4}{3}N_m ;$$

$$\sum_{n=1}^{\infty} \frac{(2m-1)_i B_n}{\pi[(2m-1)^2 - 4n^2]} + A_m\left[\frac{1}{4} + \frac{4}{\pi^2(2m-1)^2}\right] + \tag{2.12}$$

$$+\left[\sum_{n=1}^{\infty} \pi(2n-1)A_n - h\nu_0(0)\right]\left[\frac{3}{2\pi(2m-1)} -\right.$$

$$\left. - \frac{8}{\pi^3(2m-1)^3}\left(\frac{1}{4} + \frac{4}{\pi^2(2m-1)^2}\right)\right] = \frac{1}{3}S_m ;$$

$$N_m = \frac{1}{h^2}\iint_{\Omega_1} f\psi_{2m}dS ; \quad S_m = \frac{1}{h^2}\iint_{\Omega_1} f\varphi_{2m-1}dS .$$

The solution of the system in (2.12) can be found, if in all the sums of expressions (2.9), (2.10), and (2.12) we restrict ourselves to the finite number of the terms N_0 which depends on the required accuracy of calculations. Then from (2.12) we have $2N_0$ equations for A_m and B_m, where $m = 1, 2, ..., N_0$.

In conclusion, it should be noted that due to the triple symmetry the obtained solutions of the first and second boundary-value problems of heat conduction (1.12) and (1.9) are simultaneously the solution of the corresponding problems also for the region of a regular triangle with sides $h/3$ and $h/\sqrt{3}$ and hypotenuse $2h/3$, with the boundary conditions of heat insulation being set on the smaller side and hypotenuse and the Dirichlet or Neumann conditions on the larger side.

NOTATION

h, height of the regular triangle; ξ_i, dimensionless variables; $\mu(t)$, $p(t)$, and $\nu_0(T)$, functions; A_n and B_n, generalized Fourier coefficients; W_1, E_1, T_j, V, S_m, and N_n, auxiliary functions; $\{\varphi_n, \psi_n\}$, functional base.

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